On the Cauchy problem for higher-order nonlinear dispersive equations

Didier Pilod

Abstract

We study the higher-order nonlinear dispersive equation

$$\partial_t u + \partial_x^{2j+1} u = \sum_{0 \le j_1 + j_2 \le 2j} a_{j_1, j_2} \partial_x^{j_1} u \partial_x^{j_2} u, \quad x, \ t \in \mathbb{R}.$$

where u is a real- (or complex-) valued function. We show that the associated initial value problem is well posed in weighted Besov and Sobolev spaces for small initial data. We also prove ill-posedness results when $a_{0,k} \ne 0$ for some $k > j$, in the sense that this equation cannot have its flow map C^2 at the origin in $H^s(\mathbb{R})$, for any $s \in \mathbb{R}$. The same technique leads to similar ill-posedness results for other higher-order nonlinear dispersive equation as higher-order Benjamin-Ono and intermediate long wave equations.

1   Introduction

In this paper we consider the initial value problem (IVP)

$$\begin{cases} \partial_t u + \partial_x^{2j+1} u = \sum_{0 \le j_1 + j_2 \le 2j} a_{j_1, j_2} \partial_x^{j_1} u \partial_x^{j_2} u, \quad x, \ t \in \mathbb{R}, \\ u(0) = \phi, \end{cases} \tag{1}$$

where u is a real- (or complex-) valued function and a_{l_1, l_2} are constants in $\mathbb{R}$ or $\mathbb{C}$. It is a particular case of the class of IVPs

$$\begin{cases} \partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \dots, \partial_x^{2j} u), \quad x, \ t \in \mathbb{R}, \ j \in \mathbb{N} \\ u(0) = u_0, \end{cases} \tag{2}$$

where

$$P : \mathbb{R}^{2j+1} \to \mathbb{R} \quad (\text{or } P : \mathbb{C}^{2j+1} \to \mathbb{C})$$

is a polynomial having no constant or linear terms.

The class of IVPs (2) contains the KdV hierarchy as well as higher-order models in water waves problems (see [9] for the references). When $j = 1$ and the nonlinearity has the form $u\partial_x u$, the equation (1) is the KdV equation, when $j = 1$ and the nonlinearity has the form $\alpha(\partial_x u)^2 + \gamma u \partial_x^2 u$, it becomes the limit (when the dissipation tends to zero) of the KdV-Kuramoto-Velarde (KdV-KV) equation (see [1] and [15]).

Kenig, Ponce and Vega have proved that the class of IVPs (2) is well-posed in some weighted Sobolev spaces for small initial data [8], and for arbitrary initial data [9]. In [1] Argento found the best exponents of the weighted Sobolev spaces where well-posedness for the non dissipative KdV-KV equation is satisfied. More precisely, she showed that this IVP is well-posed for small initial data in $H^k(\mathbb{R}) \cap H^3(\mathbb{R}; x^2 dx)$ for $k \in \mathbb{N}$, $k \geq 5$.

The method used, in the case of small initial data, is an application of a fixed point theorem to the associated integral equation, taking advantage of the smoothing effects associated to the unitary group of the linear equation. In particular, a maximal (in time) function estimate is needed in L_x^1. Actually, as observed in [7], the L_x^1-maximal function estimate fails without weight. In the case of arbitrary initial data, Kenig, Ponce and Vega performed a gauge transformation on the equation (2) to get a dispersive system whose nonlinear terms are independent of the higher-order derivative. This allows to apply the techniques already used in the case of small initial data.

In the following, we improved these results for the IVP (1) in the case of small initial data, using weighted Besov spaces. The use of Besov spaces is inspired by the works of Molinet and Ribaud on the Korteweg-de Vries equation [13] and on the Benjamin-Ono equation [14], and of Planchon on the nonlinear Schrödinger equation [16]. It allows to refine the L_x^1-maximal function estimate, using the L_x^4-maximal function estimate derived by Kenig and Ruiz [10] (see also [5]), and to obtain well-posedness results in fractional weighted Besov spaces.

Nevertheless, the natural spaces to show well-posedness for the equation (1) are the Sobolev spaces $H^s(\mathbb{R})$. We prove here that if there exists $k > j$ such that $a_{0,k} \neq 0$, we cannot solve this problem in any space continuously embedded in $C([-T, T]; H^s(\mathbb{R}))$, for any $s \in \mathbb{R}$, using a fixed point theorem on the integral equation. As a consequence of this result, we deduce that in this case, the flow-map data solution of (1) cannot be C^2 at the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$, for any $s \in \mathbb{R}$.

The same kind of argument leads to similar results for other higher-order nonlinear dispersive equations. We consider first a higher order Benjamin-

Ono equation.

$$\begin{cases} \partial_t u - bH\partial_x^2 u + a\epsilon\partial_x^3 u = cu\partial_x u - d\epsilon\partial_x(uH\partial_x u + H(u\partial_x u)) \\ u(0) = \phi, \end{cases} \quad (3)$$

where H is the Hilbert transform, u is a real-valued function, and $a \in \mathbb{R}$, b, c and d are positive constants. This equation was derived by Craig, Guyenne and Kalisch [3], using a Hamiltonian perturbation theory. It describes, as the Benjamin-Ono equation, the evolution of weakly nonlinear dispersive internal long waves at the interface of a two-layer system, one being infinitely deep.

In [3], Craig, Guyenne and Kalisch (always using a Hamiltonian perturbation theory) also derived a higher order intermediate long wave equation.

$$\begin{cases} \partial_t u - b\mathcal{F}_h\partial_x^2 u + (a_1\mathcal{F}_h^2 + a_2)\epsilon\partial_x^3 u = cu\partial_x u - d\epsilon\partial_x(u\mathcal{F}_h\partial_x u + \mathcal{F}_h(u\partial_x u)) \\ u(0) = \phi, \end{cases}$$

$$(4)$$

where $\mathcal{F}_h$ is the Fourier multiplier $-i\coth(h\xi)$, u is a real-valued solution, and a_1, a_2, b, c, d and h are positive constants. The same ill-posedness results also apply for these equations.

These results are inspired by those from Molinet, Saut and Tzvetkov for the KPI equation [11] and the Benjamin-Ono (and the ILW) equation [12], (see also Bourgain [2] and Tzvetkov [20] for the KdV equation). It is worth notice that the equation (3) and the BO equation (as well as the equation (4) and the ILW equation) share the same property of ill-posedness of the flow in any Sobolev space $H^s(\mathbb{R})$.

The rest of this paper is organized as follows: in Section 2, we introduce a few notation, define the function spaces and state our main results. In Section 3, we derive some linear estimates that we use in Section 4 to prove our well-posedness results. Finally, in Section 5, we deal with the ill-posedness results.

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2   Statements of the results

1. Some notations. For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$. And we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

Let $U_j(t) = e^{-t\partial_x^{2j+1}}$ be the unitary group (in $H^s(\mathbb{R})$) associated to the Airy equation, so that we have *via* Fourier transform

$$U_j(t)\phi = \left(e^{(-1)^{j+1} i \xi^{2j+1} t} \widehat{\phi} \right)^{\vee}, \quad \forall\, t \in \mathbb{R}, \quad \forall \phi \in H^s(\mathbb{R}). \tag{5}$$

The group U_j commute with the operator of multiplication by x.

Lemma 1 *Let $j \geq 1$ and $f \in \mathcal{S}(\mathbb{R})$, then we have*

$$xU_j(t)f = U_j(t)(xf) + (2j+1)tU_j(t)\partial_x^{2j} f \quad \forall\, t \in \mathbb{R}. \tag{6}$$

Proof. see [17].

2. Littlewood-Paley multipliers. Throughout the paper, we fix a cutoff function χ such that

$$\chi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \chi \leq 1, \quad \chi_{|_{[-1,1]}} = 1 \quad \text{and} \quad \mathrm{supp}(\chi) \subset [-2, 2]. \tag{7}$$

We define

$$\psi(\xi) := \chi(\xi) - \chi(2\xi) \quad \text{and} \quad \psi_l(\xi) := \psi(2^{-l}\xi), \tag{8}$$

so that we have

$$\sum_{l \in \mathbb{Z}} \psi_l(\xi) = 1, \ \forall \xi \neq 0 \quad \text{and} \quad \mathrm{supp}(\psi_l) \subset \{2^{l-1} \leq |\xi| \leq 2^{l+1}\}. \tag{9}$$

Next, we define the Littlewood-Paley multipliers by

$$\Delta_l f = \left(\psi_l \widehat{f} \right)^{\vee} = (\psi_l)^{\vee} * f \quad \forall f \in \mathcal{S}'(\mathbb{R}), \ \forall l \in \mathbb{Z}, \tag{10}$$

and

$$S_l f = \sum_{k \leq l} \Delta_k f \quad \forall f \in \mathcal{S}'(\mathbb{R}), \ \forall l \in \mathbb{Z}. \tag{11}$$

More precisely we have that

$$S_0 f = \left(\chi \widehat{f} \right)^{\vee} \quad \forall f \in \mathcal{S}'(\mathbb{R}), \tag{12}$$

This means that S_0 is the operator of restriction in the low frequencies. Note also that since $(\psi_l)^\vee = 2^l(\psi)^\vee(2^l\cdot)$, $\|(\psi_l)^\vee\|_{L^1} = C$ and then, by Young's inequality we have that for all $l \in \mathbb{Z}$

$$\|\Delta_l f\|_{L^p} \leq C\|f\|_{L^p}, \ \forall \ f \in L^p, \ \forall p \in [1, +\infty]. \tag{13}$$

Combining this result with the integral Minkowski inequality, we also deduce that

$$\|\Delta_l f\|_{L_x^p L_t^q} \leq C\|f\|_{L_x^p L_t^q}, \ \forall \ f \in L_x^p L_t^q, \ \forall p, \ q \in [1, +\infty]. \tag{14}$$

We will need to commute S_0 and Δ_l with the operator of multiplication by x

$$[S_0, x]f = S_0'f \quad \text{where} \quad S_0'f = \left(\left(\frac{d}{d\xi}\chi\right)\widehat{f}\right)^\vee \tag{15}$$

$$[\Delta_l, x]f = \Delta_l'f \quad \text{where} \quad \Delta_l'f = \left(2^{-l}\left(\frac{d}{d\xi}\psi\right)(2^{-l}\cdot)\widehat{f}\right)^\vee \tag{16}$$

Finally, let $\tilde{\psi}$ be another smooth function supported in $\mathbb{R} \setminus \{0\}$ such that $\tilde{\psi} = 1$ on supp(ψ). We define $\tilde{\Delta}_l$ like Δ_l with $\tilde{\psi}$ instead of ψ which yields in particular the following identity

$$\tilde{\Delta}_l\Delta_l = \Delta_l. \tag{17}$$

3. Function spaces. Let $1 \leq p, q \leq \infty$, $T > 0$, the mixed "space-time" Lebesgue spaces are defined by

$$L_x^p L_T^q := \{u : \mathbb{R} \times [-T, T] \to \mathbb{R} \text{ measurable } : \ \|u\|_{L_x^p L_T^q} < \infty\},$$

and

$$L_T^q L_x^p := \{u : \mathbb{R} \times [-T, T] \to \mathbb{R} \text{ measurable } : \ \|u\|_{L_T^q L_x^p} < \infty\},$$

where

$$\|u\|_{L_x^p L_T^q} := \left(\int_{\mathbb{R}} \|u(x, \cdot)\|_{L^q([-T,T])}^p dx\right)^{1/p}, \tag{18}$$

and

$$\|u\|_{L_T^q L_x^p} := \left(\int_{-T}^T \|u(\cdot, t)\|_{L^p(\mathbb{R})}^q dt\right)^{1/q}. \tag{19}$$

Next we derive the following Berstein's inequalities.

Lemma 2 *Let* $f : \mathbb{R} \times [0, T] \to \mathbb{C}$ *a smooth function and* $p, \ q \in [1, +\infty]$, *then we have for all* $j, \ l \in \mathbb{N}$,

$$\|\Delta_l \partial_x^j f\|_{L_x^p L_T^q} \lesssim 2^{jl} \|\Delta_l f\|_{L_x^p L_T^q}, \tag{20}$$

and

$$\|x \Delta_l \partial_x^j f\|_{L_x^p L_T^q} \lesssim 2^{jl} \|x \Delta_l f\|_{L_x^p L_T^q} + 2^{j(l-1)} \|\Delta_l f\|_{L_x^p L_T^q}. \tag{21}$$

Proof. We deduce from (17), (18), integral Minkowski's and Young's inequalities that

$$\begin{aligned}
\|\Delta_l \partial_x^j f\|_{L_x^p L_T^q} &= \|\tilde{\Delta}_l \Delta_l \partial_x^j f\|_{L_x^p L_T^q} = \| \|\partial_x^j (\tilde{\psi}_l)^\vee * \Delta_l f(\cdot, t)\|_{L_T^q} \|_{L_x^p} \\
&\leq \| \int_{\mathbb{R}} |\partial_x^j (\tilde{\psi}_l)^\vee (y)| \|\Delta_l f(x - y, t)\|_{L_T^q} dy \|_{L_x^p} \\
&\lesssim \|\partial_x^j (\tilde{\psi}_l)^\vee\|_{L_x^1} \|\Delta_l f\|_{L_x^p L_T^q}.
\end{aligned}$$

This implies (20), since $\|\partial_x^j (\tilde{\psi}_l)^\vee\|_{L_x^1} = c 2^{lj}$. By a similar argument, we have

$$\begin{aligned}
\|x \Delta_l \partial_x^j f\|_{L_x^p L_T^q} &\leq \|x \left(|\partial_x^j (\tilde{\psi}_l)^\vee| * \|\Delta_l f(\cdot, t)\|_{L_T^q} (x) \right) \|_{L_x^p} \\
&\lesssim \|\partial_x^j (\tilde{\psi}_l)^\vee\|_{L_x^1} \|x \Delta_l f\|_{L_x^p L_T^q} + \|x \partial_x^j (\tilde{\psi}_l)^\vee\|_{L_x^1} \|\Delta_l f\|_{L_x^p L_T^q},
\end{aligned}$$

which implies inequality (21), since $\|\partial_x^j (\tilde{\psi}_l)^\vee\|_{L_x^1} = c 2^{lj}$ and $\|x \partial_x^j (\tilde{\psi}_l)^\vee\|_{L_x^1} = c 2^{(l-1)j}$. $\qquad \square$

We will also use the fractional Sobolev spaces. Let $s \in \mathbb{R}$, then

$$H^s(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) \ : \ (1 + \xi^2)^{\frac{s}{2}} \widehat{f}(\xi) \in L^2(\mathbb{R})\}$$

with the norm

$$\|f\|_{H^s} := \|(1 + \xi^2)^{s/2} \widehat{f}(\xi)\|_{L^2}. \tag{22}$$

When $s = k \in \mathbb{N}$, it is well known (see for example [18]) that

$$H^k(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \ : \ \partial_x^l f \in L^2(\mathbb{R}), \ l = 0, 1 \cdots k\},$$

with the equivalent norm

$$\|f\|_{L_k^2} := \sum_{l=0}^{k} \|\partial_x^l f\|_{L^2} \sim \|f\|_{H^k}. \tag{23}$$

Similarly, it is possible to define weighted Sobolev spaces. Let $k \in \mathbb{N}$, then

$$H^k(\mathbb{R}; x^2 dx) := \{f \in L^2(\mathbb{R}; x^2 dx) \ : \ \partial_x^l f \in L^2(\mathbb{R}; x^2 dx), \ l = 0, 1 \cdots k\},$$

with the norm

$$\|f\|_{H^k(x^2 dx)} := \sum_{l=0}^{k} \|x\partial_x^l f\|_{L^2}. \tag{24}$$

Finally, we recall the definition of the Besov spaces and define weighted Besov spaces. Let $s \in \mathbb{R}$, $p, q \geq 1$, the non homogeneous Besov space $\mathcal{B}_p^{s,q}(\mathbb{R})$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R})$ under the norm

$$\|f\|_{\mathcal{B}_p^{s,q}} := \|S_0 f\|_{L^p} + \|\{2^{ls} \|\Delta_l f\|_{L^p}\}_{l \geq 0}\|_{l^q(\mathbb{N})}. \tag{25}$$

This definition naturally extends (even if $s \in \mathbb{R}$) for weighted spaces. Let $s \in \mathbb{R}$, p, $q \geq 1$, then $\mathcal{B}_p^{s,q}(\mathbb{R}; x^p dx)$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R})$ under the norm

$$\|f\|_{\mathcal{B}_p^{s,q}(x^p dx)} := \|x S_0 f\|_{L^p} + \|\{2^{ls} \|x \Delta_l f\|_{L^p}\}_{l \geq 0}\|_{l^q(\mathbb{N})}. \tag{26}$$

It is well known (see [19]) that for all $s \in \mathbb{R}$

$$H^s(\mathbb{R}) = \mathcal{B}_2^{s,2}(\mathbb{R}) \quad \text{and that} \quad \|f\|_{H^s} \sim \|f\|_{\mathcal{B}_2^{s,2}}. \tag{27}$$

Next we derive a similar result for weighted spaces in the case $s = k \in \mathbb{N}$.

Lemma 3 *Let $k \in \mathbb{N}$, $k \geq 1$ and $f \in \mathcal{S}(\mathbb{R})$, then*

$$\|f\|_{H^k(x^2 dx)} + \|f\|_{H^{k-1}} \sim \|f\|_{\mathcal{B}_2^{k,2}(x^2 dx)} + \|f\|_{H^{k-1}}. \tag{28}$$

Proof. We apply (15), (16), (27), the Plancherel theorem and the fact that the supports of $\frac{d}{d\xi}\psi(2^{-l}\xi)$ are almost disjoint to get

$$\|f\|_{\mathcal{B}_2^{k,2}(x^2 dx)} = \|x S_0 f\|_{L^2} + \left(\sum_{l \geq 0} 4^{lk} \|x \Delta_l f\|_{L^2}^2\right)^{1/2}$$

$$\leq \|S_0(xf)\|_{L^2} + \|S_0' f\|_{L^2} + \left(\sum_{l \geq 0} 4^{lk} (\|\Delta_l(xf)\|_{L^2} + \|\Delta_l' f\|_{L^2})^2\right)^{1/2}$$

$$\lesssim \|xf\|_{\mathcal{B}_2^{k,2}} + \left(\int_{\mathbb{R}} |(\frac{d}{d\xi}\chi)(\xi)\widehat{f}(\xi)|^2 d\xi + \sum_{l \geq 0} \int_{\mathbb{R}} 4^{l(k-1)} |(\frac{d}{d\xi}\psi)(2^{-l}\xi)\widehat{f}(\xi)|^2 d\xi\right)^{1/2}$$

$$\lesssim \|xf\|_{H^k} + \|\partial_x^{k-1} f\|_{L^2}.$$

Then we use (23) and the identity

$$\partial_x^l(xf) = l\partial_x^{l-1} f + x\partial_x^l f, \ \forall l \geq 1$$

to obtain that

$$\|f\|_{\mathcal{B}_2^{k,2}(x^2 dx)} \lesssim \|f\|_{H^k(x^2 dx)} + \|f\|_{H^{k-1}}. \tag{29}$$

The other inequality of (28) follows exactly by the same argument. □

4. Statements of the results.

Theorem 1 *There exists $\delta > 0$ such that for all $u_0 \in \mathcal{B}_2^{2j+1/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2 dx)$ with*

$$\beta = \|u_0\|_{\mathcal{B}_2^{2j+9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2 dx)} \leq \delta, \tag{30}$$

there exists $T = T(\beta)$ such that $T(\beta) \nearrow +\infty$ when $\beta \to 0$, a space X_T such that

$$X_T \hookrightarrow C([-T,T]; \mathcal{B}_2^{2j+1/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2 dx)) \tag{31}$$

and a unique solution u of (1) in X_T. Moreover, the flow map is smooth from $\mathcal{B}_2^{2j+1/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2 dx)$ to X_T near the origin.

Theorem 2 *Let $s > 2j + 1/4$, then there exists $\delta > 0$ such that for all $u_0 \in H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2j,2}(\mathbb{R}; x^2 dx)$ with*

$$\beta = \|u_0\|_{\mathcal{B}_2^{2j+1/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2 dx)} \leq \delta, \tag{32}$$

there exists $T = T(\beta)$ such that $T(\beta) \nearrow +\infty$ when $\beta \to 0$, a space $Y_{T,s}$ such that

$$Y_{T,s} \hookrightarrow C([-T,T]; H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2j,2}(\mathbb{R}; x^2 dx)) \tag{33}$$

and a unique solution u of (1) in $Y_{T,s}$. Moreover, the flow map is smooth from $H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2j,2}(\mathbb{R}; x^2 dx)$ to $Y_{T,s}$ near the origin.

Corollary 1 *Let $k \in \mathbb{N}$ such that $k > 2j + 1/4$, then the IVP (1) is locally well-posed in the space $H^k(\mathbb{R}) \cap H^{k-2j}(\mathbb{R}; x^2 dx)$ for small initial data.*

Proof. We know by Lemma 3, that $H^k(\mathbb{R}) \cap \mathcal{B}_2^{k-2j,2}(\mathbb{R}; x^2 dx) = H^k(\mathbb{R}) \cap H^{k-2j}(\mathbb{R}; x^2 dx)$, then Corollary 1 follows directly from Theorem 2. □

Remark 1 *Corollary 1 improves the previous results in [1] for the non dissipative KdV-KV equation.*

Moreover, we have the following ill-posedness results for the IVP (1).

Theorem 3 *Let $s \in \mathbb{R}$ and $T > 0$, suppose that there exists $k > j$ such that $a_{0,k} \neq 0$, then, there does not exist any space X_T such that X_T is continuously embedded in $C([-T,T]; H^s(\mathbb{R}))$, i.e.*

$$\|u\|_{C([-T,T];H^s)} \lesssim \|u\|_{X_T}, \quad \forall\, u \in X_T, \tag{34}$$

and such that

$$\|U_j(t)\phi\|_{X_T} \lesssim \|\phi\|_{H^s}, \quad \forall\, \phi \in H^s(\mathbb{R}), \tag{35}$$

and, for all u, $v \in X_T$,

$$\|\int_0^t U_j(t - t') \sum_{0 \leq l_1 \leq l_2 \leq 2j} a_{l_1,l_2} \partial_x^{l_1} u(t') \partial_x^{l_2} v(t') dt'\|_{X_T} \lesssim \|u\|_{X_T} \|v\|_{X_T}. \tag{36}$$

Theorem 4 *Let $s \in \mathbb{R}$, suppose that there exists $k > j$ such that $a_{0,k} \neq 0$. Then, if the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{R})$, the flow map data-solution*

$$S(t) : H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R}), \quad \phi \longmapsto u(t) \tag{37}$$

is not C^2 at zero.

These ill-posedness results also apply for the higher-order equations (3) and (4)

Theorem 5 *Let $s \in \mathbb{R}$. If the Cauchy problems (3) and respectively (4) are locally well-posed in $H^s(\mathbb{R})$, then the associated flow maps data-solution*

$$S^{hoBO}(t) : H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R}), \quad \phi \longmapsto u(t), \tag{38}$$

and respectively

$$S^{hoILW}(t) : H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R}), \quad \phi \longmapsto u(t) \tag{39}$$

are not C^2 at zero.

3 Linear estimates

1. Linear estimates for the free and the non homogeneous evolutions.

Proposition 1 (Kato type smoothing effect.) *Let $j \geq 1$. If $u_0 \in L^2(\mathbb{R})$, then*

$$\|\partial_x^j U_j(t)u_0\|_{L_x^\infty L_t^2} \lesssim \|u_0\|_{L^2}. \tag{40}$$

Let $T > 0$, then if $f \in L_x^1 L_T^2$

$$\|\int_0^t \partial_x^j U_j(t-t')f(\cdot,t')dt'\|_{L_T^\infty L_x^2} \lesssim \|f\|_{L_x^1 L_T^2}, \tag{41}$$

and

$$\|\int_0^t \partial_x^{2j} U_j(t-t')f(\cdot,t')dt'\|_{L_x^\infty L_T^2} \lesssim \|f\|_{L_x^1 L_T^2}. \tag{42}$$

Proof. See [9].

Proposition 2 (Maximal function estimate.) *If $u_0 \in \mathcal{S}(\mathbb{R})$, then*

$$\|U_j(t)u_0\|_{L_x^4 L_t^\infty} \lesssim \|D_x^{1/4} u_0\|_{L^2}, \tag{43}$$

and

$$\|U_j(t)u_0\|_{L_x^1 L_T^\infty} \lesssim \|D_x^{1/4} u_0\|_{L^2} + \|D_x^{1/4}(xu_0)\|_{L^2} + T\|D_x^{1/4}\partial_x^{2j} u_0\|_{L^2}. \tag{44}$$

Proof. The estimate (43) is due to Kenig, Ponce and Vega [5] (see also the work of Kenig and Ruiz [10] in the case $j = 1$). We will prove the estimate (44) using (6), (43) and Hölder's inequality

$$\|U_j(t)u_0\|_{L_x^1 L_T^\infty} = \int_{|x| \leq 1} \sup_{[-T,T]} |U_j(t)u_0(x)|dx + \int_{|x| > 1} \frac{1}{|x|} \sup_{[-T,T]} |xU_j(t)u_0(x)|dx$$

$$\lesssim \|U_j(t)u_0\|_{L_x^4 L_T^\infty} + \|U_j(t)(xu_0)\|_{L_x^4 L_T^\infty} + T\|U_j(t)\partial_x^{2j} u_0\|_{L_x^4 L_T^\infty}$$

$$\lesssim \|D_x^{1/4} u_0\|_{L^2} + \|D_x^{1/4}(xu_0)\|_{L^2} + T\|D_x^{1/4}\partial_x^{2j} u_0\|_{L^2}.$$

$\square$

Remark 2 *It is interesting to observe that the restriction on the s in Theorem 2 ($s > 2j + 1/4$) appears in the estimate (44).*

2. Linear estimates for phase localized functions. Following the ideas in [14], we will derive linear estimates for the phase localized free and nonhomogeneous evolutions.

Proposition 3 *Let $u_0 \in \mathcal{S}(\mathbb{R})$, then we have for all $l \geq 0$*

$$\|\Delta_l U_j(t)u_0\|_{L_T^\infty L_x^2} = \|\Delta_l u_0\|_{L_x^2}, \tag{45}$$

and

$$\|x\Delta_l U_j(t)u_0\|_{L_T^\infty L_x^2} \lesssim \|x\Delta_l u_0\|_{L_x^2} + T2^{2jl}\|\Delta_l u_0\|_{L_x^2}. \tag{46}$$

If $f : \mathbb{R} \times [0,T] \to \mathbb{C}$ is smooth, then we have for all $l \geq 0$

$$\|\int_0^t \Delta_l U_j(t-t')f(\cdot,t')dt'\|_{L_T^\infty L_x^2} \lesssim 2^{-jl}\|\Delta_l f\|_{L_x^1 L_T^2}, \tag{47}$$

and

$$\|\int_0^t x\Delta_l U_j(t-t')f(\cdot,t')dt'\|_{L_T^\infty L_x^2} \lesssim 2^{-jl}\|x\Delta_l f\|_{L_x^1 L_T^2} + T2^{jl}\|\Delta_l f\|_{L_x^1 L_T^2}. \tag{48}$$

Proof. The identity (45) follows directly from the fact that U_j is a unitary group in $L^2(\mathbb{R})$. To prove the estimate (46), we will use (6), (45) and Plancherel's theorem

$$\|x\Delta_l U_j(t)u_0\|_{L_T^\infty L_x^2} \leq \|U_j(t)(x\Delta_l u_0)\|_{L_T^\infty L_x^2} + (2j+1)T\|U_j(t)\partial_x^{2j}\Delta_l u_0\|_{L_T^\infty L_x^2}$$
$$\lesssim \|x\Delta_l u_0\|_{L_T^\infty L_x^2} + T2^{2jl}\|\Delta_l u_0\|_{L_T^\infty L_x^2}.$$

The estimate (47) follows from (41), Plancherel's theorem and the fact that Δ_l localize the frequency near $|\xi| \sim 2^l$. Next, we will prove the estimate (48). The identity (6), the estimate (41) and the fact the operator $x\Delta_l$ still localizes the frequency near $|\xi| \sim 2^l$ (see the commutator identity (16), imply that

$$\|\int_0^t x\Delta_l U_j(t-t')f(\cdot,t')dt'\|_{L_T^\infty L_x^2}$$

$$\lesssim \|\int_0^t U_j(t-t')(x\Delta_l f(\cdot,t'))dt'\|_{L_T^\infty L_x^2}$$

$$+T\|\int_0^t \Delta_l U_j(t-t')\partial_x^{2j}f(\cdot,t')dt'\|_{L_T^\infty L_x^2}$$

$$\lesssim 2^{-jl}\|x\Delta_l f\|_{L_x^1 L_T^2} + T2^{jl}\|\Delta_l f\|_{L_x^1 L_T^2}. \tag{49}$$

$\square$

Proposition 4 *Let $u_0 \in \mathcal{S}(\mathbb{R})$, then we have for all $l \geq 0$*

$$\|\Delta_l U_j(t)u_0\|_{L_x^\infty L_T^2} \lesssim 2^{-jl}\|\Delta_l u_0\|_{L_x^2}, \tag{50}$$

and

$$\|x\Delta_l U_j(t)u_0\|_{L_x^\infty L_T^2} \lesssim 2^{-jl}\|x\Delta_l u_0\|_{L_x^2} + T2^{jl}\|\Delta_l u_0\|_{L_x^2}. \tag{51}$$

If $f : \mathbb{R} \times [0,T] \to \mathbb{C}$ is smooth, then we have for all $l \geq 0$

$$\|\int_0^t \Delta_l U_j(t-t')f(\cdot,t')dt'\|_{L_x^\infty L_T^2} \lesssim 2^{-2jl}\|\Delta_l f\|_{L_x^1 L_T^2}, \tag{52}$$

and

$$\|\int_0^t x\Delta_l U_j(t-t')f(\cdot,t')dt'\|_{L_x^\infty L_T^2} \lesssim 2^{-2jl}\|x\Delta_l f\|_{L_x^1 L_T^2} + T\|\Delta_l f\|_{L_x^1 L_T^2}. \tag{53}$$

Proof. The proof is the same as for the Proposition 3 where we use (40) and (42) instead of (41). $\qquad\qquad\square$

In order to derive a non homogeneous estimate for the localized maximal function, we need the following lemma due to Molinet and Ribaud (see [14]) and inspired by a previous result of Christ and Kiselev (see [4]).

Lemma 4 *Let L be a linear operator defined on space-time functions $f(x,t)$ by*

$$Lf(t) = \int_0^T K(t,t')f(t')dt',$$

where $K : \mathcal{S}(\mathbb{R}^2) \to C(\mathbb{R}^3)$ and such that

$$\|Lf\|_{L_x^{p_1} L_T^\infty} \leq C\|f\|_{L_x^{p_2} L_T^{q_2}},$$

with $p_2, q_2 < \infty$. Then,

$$\|\int_0^t K(t,t')f(t')dt'\|_{L_x^p L_T^\infty} \leq C\|f\|_{L_x^{p_2} L_T^{q_2}}.$$

Proposition 5 *Let $u_0 \in \mathcal{S}(\mathbb{R})$, then we have for all $l \geq 0$*

$$\|\Delta_l U_j(t)u_0\|_{L_x^1 L_T^\infty} \lesssim 2^{(\frac{1}{4}+2j)l}(1+T)\|\Delta_l u_0\|_{L_x^2} + 2^{\frac{1}{4}l}\|x\Delta_l u_0\|_{L_x^2}. \tag{54}$$

If $f : \mathbb{R} \times [0,T] \to \mathbb{C}$ is smooth, then we have for all $l \geq 0$

$$\|\int_0^t \Delta_l U_j(t-t')f(\cdot,t')dt'\|_{L_x^1 L_T^\infty}$$
$$\lesssim 2^{(\frac{1}{4}-j)l}\|x\Delta_l f\|_{L_x^1 L_T^2} + (1+T)2^{(\frac{1}{4}+j)l}\|\Delta_l f\|_{L_x^1 L_T^2}. \tag{55}$$

Proof. To obtain the estimate (54), we apply (44) with $\Delta_l u_0$ instead of u_0, then we use Plancherel's theorem and the fact that the operators Δ_l and $x\Delta_l$ localize the frequency near $|\xi| \sim 2^l$.

In order to prove the estimate (55), we first need to derive a "nonretarded" L^4-maximal function estimate. Note first that duality and (43) imply that

$$\| \int_0^T \Delta_l U_j(-t) f(\cdot, t) dt \|_{L_x^2} \lesssim 2^{\frac{1}{4}l} \|\Delta_l f\|_{L_x^{4/3} L_T^1}. \tag{56}$$

Then, we deduce combining (47), (56) and the Cauchy-Schwarz inequality that for all $g \in L_x^{4/3} L_T^1$

$$\int_{\mathbb{R} \times [0,T]} \left(\int_0^T \Delta_l U_j(t - t') f(\cdot, t') dt' \right) g(x, t) dx dt$$

$$= \int_{\mathbb{R}} \left(\int_0^T U_j(-t') \Delta_l f(\cdot, t') dt' \right) \left(\int_0^T U_j(-t) \tilde{\Delta}_l \overline{g}(\cdot, t) dt \right) dx$$

$$\leq \| \int_0^T U_j(-t') \Delta_l f(\cdot, t') dt' \|_{L_x^2} \| \int_0^T U_j(-t) \tilde{\Delta}_l \overline{g}(\cdot, t) dt \|_{L_x^2}$$

$$\lesssim 2^{-jl} \|\Delta_l f\|_{L_x^1 L_T^2} 2^{\frac{1}{4}l} \|g\|_{L_x^{4/3} L_T^1},$$

so that by duality

$$\| \int_0^T \Delta_l U_j(t - t') f(\cdot, t') dt' \|_{L_x^4 L_T^\infty} \lesssim 2^{(\frac{1}{4} - j)l} \|\Delta_l f\|_{L_x^1 L_T^2}. \tag{57}$$

Then, we use Lemma 4 to obtain the corresponding "retarded" estimate

$$\| \int_0^t \Delta_l U_j(t - t') f(\cdot, t') dt' \|_{L_x^4 L_T^\infty} \lesssim 2^{(\frac{1}{4} - j)l} \|\Delta_l f\|_{L_x^1 L_T^2}. \tag{58}$$

We are now able to derive the $L_x^1 L_T^\infty$ estimate for the non homogeneous term. We have by Hölder's inequality

$$\| \int_0^t \Delta_l U_j(t - t') f(\cdot, t') dt' \|_{L_x^1 L_T^\infty}$$

$$= \int_{|x| \leq 1} \sup_{t \in [-T,T]} \left| \int_0^t \Delta_l U_j(t - t') f(\cdot, t') dt' \right| dx$$

$$+ \int_{|x| > 1} \frac{1}{|x|} \sup_{t \in [-T,T]} \left| \int_0^t x\Delta_l U_j(t - t') f(\cdot, t') dt' \right| dx \tag{59}$$

$$\lesssim \| \int_0^t \Delta_l U_j(t - t') f(\cdot, t') dt' \|_{L_x^4 L_T^\infty} + \| \int_0^t x\Delta_l U_j(t - t') f(\cdot, t') dt' \|_{L_x^4 L_T^\infty}$$

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Thus, we deduce from (6), (58) and (59) that

$$\| \int_0^t \Delta_l U_j(t - t')f(\cdot, t')dt'\|_{L_x^1 L_T^\infty}$$
$$\lesssim (2^{(\frac{1}{4}-j)l} + T2^{(\frac{1}{4}+j)l})\|\Delta_l\|_{L_x^1 L_T^2} + 2^{(\frac{1}{4}-j)l}\|x\Delta_l f\|_{L_x^1 L_T^2}, \qquad (60)$$

which leads to (55), since $l \geq 0$. $\qquad \square$

Remark 3 *All the results in Propositions 3, 4 and 5 are still valid with S_0 instead of Δ_l and $l = 0$.*

4 Proof of Theorems 1 and 2

Proof of Theorem 1. Consider the integral equation associated to (1)

$$u(t) = F(u)(t), \qquad (61)$$

where

$$F(u)(t) := U_j(t)u_0 + \int_0^t U_j(t - t') \sum_{0 \leq j_1 + j_2 \leq 2j} a_{j_1, j_2} \partial_x^{j_1} u(t') \partial_x^{j_2} u(t')dt'. \qquad (62)$$

Let $T > 0$, define the following seminorms:

$$N_1^T(u) = \|S_0 u\|_{L_T^\infty L_x^2} + \sum_{l=1}^\infty 2^{(2j+\frac{1}{4})l}\|\Delta_l u\|_{L_T^\infty L_x^2}, \qquad (63)$$

$$N_2^T(u) = \|x S_0 u\|_{L_T^\infty L_x^2} + \sum_{l=1}^\infty 2^{\frac{1}{4}l}\|x\Delta_l u\|_{L_T^\infty L_x^2}, \qquad (64)$$

$$P_1^T(u) = \|S_0 u\|_{L_x^\infty L_T^2} + \sum_{l=1}^\infty 2^{(3j+\frac{1}{4})l}\|\Delta_l u\|_{L_x^\infty L_T^2}, \qquad (65)$$

$$P_2^T(u) = \|x S_0 u\|_{L_x^\infty L_T^2} + \sum_{l=1}^\infty 2^{(j+\frac{1}{4})l}\|x\Delta_l u\|_{L_x^\infty L_T^2}, \qquad (66)$$

$$M^T(u) = \|S_0 u\|_{L_x^1 L_T^\infty} + \sum_{l=1}^\infty \|\Delta_l u\|_{L_x^1 L_T^\infty}. \qquad (67)$$

Then, we define the Banach space

$$X_T = \{u \in C([-T, T]; \mathcal{B}_2^{2j+1/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2 dx)) \; : \; \|u\|_{X_T} < \infty\}, \qquad (68)$$

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where
$$\|u\|_{X_T} = N_1^T(u) + N_2^T(u) + P_1^T(u) + P_2^T(u) + M^T(u). \qquad (69)$$

We deduce from (45), (46), (50), (51) and (54) that

$$\|U_j(t)u_0\|_{X_T} \lesssim (1+T)\left(\|u_0\|_{\mathcal{B}_2^{2j+1/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2\,dx)}\right), \qquad (70)$$

and from (47), (48), (52), (53) and (55) that

$$\begin{aligned}
\|\int_0^t &U_j(t-t') \sum_{0 \le j_1+j_2 \le 2j} a_{j_1,j_2} \partial_x^{j_1} u(t') \partial_x^{j_2} v(t')dt'\|_{X_T} \\
&\lesssim (1+T) \sum_{0 \le j_1+j_2 \le 2j} |a_{j_1,j_2}| \left(\|S_0(\partial_x^{j_1} u \partial_x^{j_2} v)\|_{L_x^1 L_T^2}\right. \\
&\quad + \sum_{l=1}^{\infty} 2^{(j+\frac{1}{4})l}\|\Delta_l(\partial_x^{j_1} u \partial_x^{j_2} v)\|_{L_x^1 L_T^2} + \|x S_0(\partial_x^{j_1} u \partial_x^{j_2} v)\|_{L_x^1 L_T^2} \\
&\quad \left. + \sum_{l=1}^{\infty} 2^{(\frac{1}{4}-j)l}\|x\Delta_l(\partial_x^{j_1} u \partial_x^{j_2} v)\|_{L_x^1 L_T^2}\right). \qquad (71)
\end{aligned}$$

In order to estimate the nonlinear term $\sum_{l=1}^{\infty} 2^{(j+\frac{1}{4})l}\|\Delta_l(\partial_x^{j_1} u \partial_x^{j_2} v)\|_{L_x^1 L_T^2}$, we observe that

$$\begin{aligned}
\Delta_l(fg) &= \Delta_l\left((S_0 f + \sum_{r \ge 1}\Delta_r f)(S_0 g + \sum_{k \ge 1}\Delta_k g)\right) \\
&= \Delta_l\left(S_0 f S_0 g + \sum_{r \ge 1}\Delta_r f S_r g + \sum_{r \ge 1}\Delta_r g S_{r-1} f\right), \qquad (72)
\end{aligned}$$

where $f = \partial_x^{j_1} u$ and $g = \partial_x^{j_2} v$. First, since $\Delta_l(S_0 u S_0 v) = 0$ for $l \ge 3$ and since the operators Δ_l are uniformly bounded (in l) in L^1, we have by Hölder's inequality

$$\sum_{l \ge 1} 2^{(j+\frac{1}{4})l}\|\Delta_l(S_0\partial_x^{j_1} u S_0 \partial_x^{j_2} v)\|_{L_x^1 L_T^2} \lesssim \|S_0 u\|_{L_x^\infty L_T^2}\|S_0 v\|_{L_x^1 L_T^\infty}. \qquad (73)$$

In order to estimate the second term on the right-hand side of (72), we notice, since the term $\Delta_r \partial_x^{j_1} u S_r \partial_x^{j_2} v$ is localized in frequency in the set $|\xi| \le 2^{r+3}$ and the operator Δ_l only sees the frequency in the set $2^{l-1} \le |\xi| \le 2^{l+1}$, that

$$\Delta_l\left(\sum_{r=1}^{\infty}\Delta_r f S_r g\right) = \Delta_l\left(\sum_{r \ge l-3}\Delta_r f S_r g\right). \qquad (74)$$

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Then, we only have to estimate terms of the form $\Delta_l(\sum_{r\geq l}\Delta_r\partial_x^{j_1}uS_r\partial_x^{j_2}v)$. Using Hölder's inequality, the estimate (20), and the fact that

$$\|S_r\partial_x^{j_2}v\|_{L_x^1L_T^\infty} \leq \|S_0v\|_{L_x^1L_T^\infty} + \sum_{k=1}^r \|\Delta_k\partial_x^{j_2}v\|_{L_x^1L_T^\infty}$$

$$\leq \|S_0v\|_{L_x^1L_T^\infty} + \sum_{k=1}^r 2^{j_2k}\|\Delta_kv\|_{L_x^1L_T^\infty} \lesssim 2^{j_2r}M^T(v), \quad (75)$$

we deduce that

$$\sum_{l\geq 1}2^{(j+\frac{1}{4})l}\|\Delta_l(\sum_{r\geq l}\Delta_r\partial_x^{j_1}uS_r\partial_x^{j_2}v)\|_{L_x^1L_T^2}$$

$$\leq \sum_{l\geq 1}2^{(j+\frac{1}{4})l}\sum_{r\geq l}\|\Delta_r\partial_x^{j_1}u\|_{L_x^\infty L_T^2}\|S_r\partial_x^{j_2}v\|_{L_x^1L_T^\infty}$$

$$\leq M^T(v)\sum_{r\geq 1}\left(\sum_{l=1}^r2^{(j+\frac{1}{4})l}\right)2^{(j_1+j_2)r}\|\Delta_ru\|_{L_x^\infty L_T^2}$$

$$\lesssim M^T(v)P_1^T(u) \leq \|u\|_{X_T}\|v\|_{X_T}, \quad (76)$$

since $j_1+j_2\leq 2j$. Thus, we obtain, gathering (72), (73), (74) and (76) that

$$\sum_{l=1}^\infty 2^{(j+\frac{1}{4})l}\|\Delta_l(\partial_x^{j_1}u\partial_x^{j_2}v)\|_{L_x^1L_T^2} \lesssim \|u\|_{X_T}\|v\|_{X_T}. \quad (77)$$

We apply exactly the same strategy to estimate the other nonlinear term $\sum_{l=1}^\infty 2^{(\frac{1}{4}-j)l}\|x\Delta_l(\partial_x^{j_1}u\partial_x^{j_2}v)\|_{L_x^1L_T^2}$. Then, we have only to estimate terms of the form

$$\sum_{l=1}^\infty 2^{(\frac{1}{4}-j)l}\|x\Delta_l(\sum_{r\geq l}\Delta_r\partial_x^{j_1}uS_r\partial_x^{j_2}v)\|_{L_x^1L_T^2}.$$

For this, we combine the same ideas as for the estimate (77) with the commutator identity (16) and the fact that the operators Δ_l' are also uniformly bounded (in l) in L^1 to deduce that

$$\sum_{l\geq 1}2^{(\frac{1}{4}-j)l}\|x\Delta_l(\partial_x^{j_1}u\partial_x^{j_2}v)\|_{L_x^1L_T^2} \lesssim \sum_{l\geq 1}2^{(\frac{1}{4}-j)l}\sum_{r\geq l}\|x\Delta_r\partial_x^{j_1}uS_r\partial_x^{j_2}v\|_{L_x^\infty L_T^2}$$

$$+ \sum_{l\geq 1}2^{-(\frac{3}{4}+j)l}\sum_{r\geq l}\|\Delta_r\partial_x^{j_1}uS_r\partial_x^{j_1}v\|_{L_x^\infty L_T^2}$$

$$\lesssim M^T(v)(P_1^T(u) + P_2^T(u)). \quad (78)$$

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Thus, we deduce from (71), (77) and (78) that

$$\| \int_0^t U_j(t-t') \sum_{j_1+j_2 \le 2j} a_{j_1,j_2} \partial_x^{j_1} u(t') \partial_x^{j_2} u(t') dt' \|_{X_T}$$

$$\lesssim (1+T)\|u\|_{X_T}\|v\|_{X_T}. \qquad (79)$$

Then, we use (70) and (79) to deduce that there exists a constant $C > 0$ such that

$$\|F(u)\|_{X_T} \le C(1+T)\left(\|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2 dx)} + \|u\|_{X_T}^2\right), \quad \forall\, u \in X_T, \qquad (80)$$

and

$$\|F(u)-F(v)\|_{X_T} \le C(1+T)(\|u\|_{X_T}+\|v\|_{X_T})\|u-v\|_{X_T}, \quad \forall\, u,v \in X_T. \qquad (81)$$

Let $X_T(a) := \{u \in X_T\ :\ \|u\|_{X_T} \le a\}$ the closed ball of X_T with radius a. $X_T(a)$ equipped with the metric induced by the norm $\|\cdot\|_{X_T}$ is a complete metric space. If we choose

$$\beta = \|u_0\|_{\mathcal{B}_2^{9/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2 dx)} \le \delta < \min\{(\frac{1}{4C})^2, 1\}, \qquad (82)$$

$$a = \sqrt{\beta}, \quad \text{and} \quad T = \frac{1}{4C\sqrt{\beta}}, \qquad (83)$$

we have that

$$2C(1+T)a < 1. \qquad (84)$$

Then, we deduce from (80) and (81) that the operator F is a contraction in $X_T(a)$ (up to the persistence property) and so, by the Picard fixed point theorem, there exists a unique solution of (61) in $X_T(a)$.

The proof of persistence, uniqueness and smoothness of the map follows by standard arguments (see for example [6]). $\qquad\square$

Proof of Theorem 2.

Lemma 5 *Let $s > 2j + 1/4$, then the injection*

$$H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2j,2}(\mathbb{R}; x^2 dx) \hookrightarrow \mathcal{B}_2^{2j+1/4,1}(\mathbb{R}) \cap \mathcal{B}_2^{1/4,1}(\mathbb{R}; x^2 dx) \qquad (85)$$

is continuous.

Proof. Let $s > 2j + 1/4$ and $f \in H^s(\mathbb{R})$. We obtain using the Cauchy-Schwarz inequality that

$$\|f\|_{\mathcal{B}_2^{2j+1/4,1}} = \|S_0 f\|_{L^2} + \sum_{l \geq 0} 2^{ls} \|\Delta_l f\|_{L^2} 2^{l(2j+1/4-s)}$$

$$\leq \|S_0 f\|_{L^2} + \left(\sum_{l \geq 1} 4^{(2j+1/4-s)l} \right)^{1/2} \left(\sum_{l \geq 1} 4^{sl} \|\Delta_l f\|_{L^2}^2 \right)^{1/2}$$

$$\lesssim \|f\|_{\mathcal{B}_2^{s,2}} \sim \|f\|_{H^s}. \tag{86}$$

Similarly, we get

$$\|f\|_{\mathcal{B}_2^{1/4,1}(x^2 dx)} \lesssim \|f\|_{\mathcal{B}_2^{s-2j,2}(x^2 dx)}, \tag{87}$$

when $s > 2j + 1/4$ and then, (86) and (87) yield (85). $\qquad\square$

Now, let $s > 2j + 1/4$. Exactly as in the proof of Theorem 1, we want to apply a fixed point theorem to solve the integral equation (61) in some good function space. In this way, define the following semi-norm

$$\|u\|_{X_{T,s}} = N_{1,s}^T(u) + N_{2,s}^T(u) + P_{1,s}^T(u) + P_{2,s}^T(u), \tag{88}$$

where

$$N_{1,s}^T(u) = \|S_0 u\|_{L_T^\infty L_x^2} + \left(\sum_{l=1}^\infty 4^{sl} \|\Delta_l u\|_{L_T^\infty L_x^2}^2 \right)^{1/2}, \tag{89}$$

$$N_{2,s}^T(u) = \|x S_0 u\|_{L_T^\infty L_x^2} + \left(\sum_{l=1}^\infty 4^{(s-2j)l} \|x \Delta_j u\|_{L_T^\infty L_x^2}^2 \right)^{1/2}, \tag{90}$$

$$P_{1,s}^T(u) = \|S_0 u\|_{L_x^\infty L_T^2} + \left(\sum_{l=1}^\infty 4^{(s+j)l} \|\Delta_l u\|_{L_x^\infty L_T^2}^2 \right)^{1/2}, \tag{91}$$

$$P_{2,s}^T(u) = \|x S_0 u\|_{L_x^\infty L_T^2} + \left(\sum_{l=1}^\infty 4^{(s-j)l} \|x \Delta_l u\|_{L_x^\infty L_T^2}^2 \right)^{1/2}. \tag{92}$$

If $u_0 \in H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2j,2}(\mathbb{R}; x^2 dx)$, by Lemma 5, it makes sense to define

$$\lambda_s := \frac{\|u_0\|_{\mathcal{B}_2^{2j+1/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2 dx)}}{\|u_0\|_{H^s} + \|u_0\|_{\mathcal{B}_2^{s-2j,2}(x^2 dx)}}. \tag{93}$$

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Then, let $Y_{T,s}$ be the Banach space

$$Y_{T,s} = \{u \in C([-T,T]; H^s(\mathbb{R}) \cap \mathcal{B}_2^{s-2j,2}(\mathbb{R}; x^2 dx)) \text{ such that } \|u\|_{Y_{T,s}} < \infty\}, \tag{94}$$

where

$$\|u\|_{Y_{T,s}} = \|u\|_{X_T} + \lambda_s \|u\|_{X_{T,s}}. \tag{95}$$

We deduce from (45), (46), (50), (51), (54), (93) and (95) that

$$\|U_j(t)u_0\|_{Y_{T,s}} \lesssim (1+T) \left(\|u_0\|_{\mathcal{B}_2^{2j+1/4,1}} + \|u_0\|_{\mathcal{B}_2^{1/4,1}(x^2 dx)} \right). \tag{96}$$

In order to estimate the nonlinear term of (62) in the norm $\|\cdot\|_{Y_{T,s}}$, we remember (79), and then it only remains to derive estimates of the form

$$\| \int_0^t U_j(t-t'))(\partial_x^{j_1} u(t') \partial_x^{j_2} u(t')) dt' \|_{X_{T,s}} \lesssim (1+T)\|u\|_{Y_{T,s}} \|v\|_{Y_{T,s}}. \tag{97}$$

In this way, we use (47), (48), (52), (53) and (55) and argue as in the proof of Theorem 1 to estimate the left-hand side of (97) by some terms of the form

$$A = \left(\sum_{l \geq 1} 4^{(s-j)l} \|\Delta_l (\sum_{r=l}^{\infty} \Delta_r \partial_x^{j_1} u S_r \partial_x^{j_2} v)\|_{L_x^1 L_T^2}^2 \right)^{1/2}, \tag{98}$$

$$B = \left(\sum_{l \geq 1} 4^{(s-3j)l} \|x\Delta_l (\sum_{r=l}^{\infty} \Delta_r \partial_x^{j_1} u S_r \partial_x^{j_2} v)\|_{L_x^1 L_T^2}^2 \right)^{1/2}, \tag{99}$$

and some others harmless terms. We next estimate A, we get from (14), Hölder's inequality, (20) and (75), the inequality

$$A \leq M^T(v) \left(\sum_{l \geq 0} 4^{(s+j)l} \left(\sum_{r=l}^{\infty} \|\Delta_r u\|_{L_x^\infty L_T^2} \right)^2 \right)^{1/2}. \tag{100}$$

Then, define

$$\gamma_r = 2^{(s+j)l} \|\Delta_r u\|_{L_x^\infty L_T^2} \quad \text{and note that} \quad \|\{\gamma_r\}_r\|_{l^2(\mathbb{N})} \leq P_{1,s}^T(u). \tag{101}$$

We deduce by (100), a change of index and Minkowski's inequality that

$$A \leq M^T(v)\|\{\sum_{r=l}^{\infty} 2^{(s+j)(l-r)}\gamma_r\}_l\|_{l^2(\mathbb{N})} = M^T(v)\|\{\sum_{k \geq 0} 2^{-(s+j)k}\gamma_{l+k}\}_l\|_{l^2(\mathbb{N})}$$

$$\leq M^T(v) \sum_{k \geq 0} 2^{-(s+j)k}\|\{\gamma_{l+k}\}_l\|_{l^2(\mathbb{N})} \leq M^T(v)\|\{\gamma_l\}_l\|_{l^2(\mathbb{N})} \sum_{k \geq 0} 2^{-(s+j)k},$$

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then (101) implies that

$$A \lesssim P_{1,s}^T(u) M^T(v). \tag{102}$$

Analogously, we obtain a similar estimate for B

$$B \lesssim P_{2,s}^T(u) M^T(v). \tag{103}$$

Thus, (102) and (103) yield (97) and we conclude the proof of Theorem 2 as for Theorem 1 using (96) and (97) instead of (70) and (79). □

Remark 4 *Unless we can use the strategy of Kenig, Ponce and Vega in [8] to show well-posedness for the IVPs (3) and (4) in weighted Sobolev spaces, it is not clear wether the technique used here applies or not.*

5   Ill-posedness results

In the proofs of Theorems 3 and 4, we will suppose, for simplicity, that the nonlinearity $\sum_{0 \leq l_1 \leq l_2 \leq 2j} a_{l_1,l_2} \partial_x^{l_1} u \partial_x^{l_2} u$ has the form $\partial_x^k(u^2)$ with $k > j$.

Proof of Theorem 3. The key point of the proof is the following algebraic relation

Lemma 6 *Let $j \in \mathbb{N}$ such that $j \geq 1$ and $\xi, \xi_1 \in \mathbb{R}$, then*

$$\xi_1^{2j+1} + (\xi - \xi_1)^{2j+1} - \xi^{2j+1} = (\xi - \xi_1) Q_{2j}(\xi, \xi_1), \tag{104}$$

where

$$Q_{2j}(\xi, \xi_1) = \sum_{l=0}^{2j} ((-1)^l C_{2j}^l - 1) \xi^{2j-l} \xi_1^l \tag{105}$$

and $C_n^l = \frac{n!}{l!(n-l)!}$.

Note that $\xi - \xi_1$ does not divide $Q_{2j}(\xi, \xi_1)$.

Let $s \in \mathbb{R}$, $k, j \in \mathbb{N}$ such that $k > j$ and $T > 0$. Suppose that there exists a space X_T such as in Theorem 3. Take $\phi, \psi \in H^s(\mathbb{R})$, and define $u(t) = U_j(t)\phi$ and $v(t) = U_j(t)\psi$. Then, we use (34), (35) and (36) to deduce that

$$\| \int_0^t U_j(t-t') \partial_x^k [(U_j(t')\phi)(U_j(t')\psi)] dt' \|_{H^s} \lesssim \|\phi\|_{H^s} \|\psi\|_{H^s}. \tag{106}$$

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We will show that (106) fails for an appropriate pair of ϕ, ψ, which would lead to a contradiction.

Define ϕ and ψ by

$$\phi = (\alpha^{-1/2}\chi_{I_1})^{\vee} \tag{107}$$

and

$$\psi = (\alpha^{-1/2}N^{-s}\chi_{I_2})^{\vee} \tag{108}$$

where

$$N \gg 1, \quad 0 < \alpha \ll 1, \quad I_1 = [\alpha/2, \alpha] \quad \text{and} \quad I_2 = [N, N + \alpha] \tag{109}$$

Note first that

$$\|\phi\|_{H^s} \sim \|\psi\|_{H^s} \sim 1. \tag{110}$$

Then, we use the algebraic relation (104), the definition of the unitary group U_j and the definition of ϕ and ψ to estimate the Fourier transform of the left-hand side of (64)

$$\left(\int_0^t U_j(t-t')\partial_x^k[(U_j(t')\phi)(U_j(t')\psi)]dt'\right)^{\wedge}(\xi)$$

$$= \int_0^t e^{(-1)^{j+1}i(t-t')\xi^{2j+1}}(i\xi)^k(e^{(-1)^{j+1}it(\cdot)^{2j+1}}\widehat{\phi}) * (e^{(-1)^{j+1}it(\cdot)^{2j+1}}\widehat{\psi})(\xi)dt'$$

$$= \int_{\mathbb{R}} e^{(-1)^{j+1}it\xi^{2j+1}}(i\xi)^k\widehat{\psi}(\xi_1)\widehat{\phi}(\xi - \xi_1)\int_0^t e^{(-1)^{j+1}it'Q_{2j}(\xi,\xi_1)(\xi-\xi_1)}dt'd\xi_1$$

$$= \int_{\mathbb{R}} e^{(-1)^{j+1}it\xi^{2j+1}}(i\xi)^k\widehat{\psi}(\xi_1)\widehat{\phi}(\xi - \xi_1)\frac{e^{(-1)^{j+1}it(\xi-\xi_1)Q_{2j}(\xi,\xi_1)} - 1}{(-1)^{j+1}i(\xi - \xi_1)Q_{2j}(\xi,\xi_1)}d\xi_1.$$

$$\sim \frac{e^{(-1)^{j+1}it\xi^{2j+1}}\xi^k}{\alpha N^s}\int_{\left\{\substack{\xi_1 \in I_2 \\ \xi - \xi_1 \in I_1}\right\}}\frac{e^{(-1)^{j+1}it(\xi-\xi_1)Q_{2j}(\xi,\xi_1)} - 1}{(\xi - \xi_1)Q_{2j}(\xi,\xi_1)}d\xi_1. \tag{111}$$

When $\xi - \xi_1 \in I_1$ and $\xi_1 \in I_2$, we have that $|(\xi - \xi_1)Q_{2j}(\xi,\xi_1)| \sim \alpha N^{2j}$. We choose $\alpha = N^{-2j-\epsilon}$, with $0 < \epsilon < 1$ so that

$$|(\xi - \xi_1)Q_{2j}(\xi,\xi_1)| \sim N^{-\epsilon} \ll 1 \tag{112}$$

and

$$\frac{e^{(-1)^{j+1}it(\xi-\xi_1)Q_{2j}(\xi,\xi_1)} - 1}{(\xi - \xi_1)Q_{2j}(\xi,\xi_1)} = ct + o(N^{-\epsilon}) \tag{113}$$

where $c \in \mathbb{C}$. We are now able to give a lower bound for the left-hand side of (106)

$$\|\int_0^t U_j(t-t')\partial_x^k[(U_j(t')\phi)(U_j(t')\psi)]dt'\|_{H^s} \gtrsim \frac{N^s}{N^s\alpha}N^k\alpha^{1/2}\alpha. \tag{114}$$

Thus we conclude from (106), (110) and (114) that

$$N^k \alpha^{1/2} = N^{k-j-\epsilon/2} \lesssim 1, \quad \forall\, N \gg 1, \tag{115}$$

which is a contradiction since $k > j$. □

Remark 5 *Since the class of equation (1) often appears in physical situations where the function u is needed to be real-valued, it is interesting to notice that Theorems 3 and 4 are also valid if we ask the functions to be real. Actually take $\phi_1 = \operatorname{Re}\phi$ and $\psi_1 = \operatorname{Re}\psi$ instead of ϕ and ψ, then*

$$\widehat{\phi_1} = \frac{\alpha^{-1/2}}{2}\chi_{\{\alpha/2 \le |\xi| \le \alpha\}} \quad and \quad \widehat{\psi_1} = \frac{\alpha^{-1/2}N^{-s}}{2}\chi_{\{N \le |\xi| \le N+\alpha\}}, \tag{116}$$

and so we can conclude the proof as above.

Proof of Theorem 4. Let $s \in \mathbb{R}$ and $k, j \in \mathbb{N}$ such that $k > j$. Suppose that there exists $T > 0$ such that the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{R})$ in the time interval $[0, T]$ and that its flow map solution $S^{j,k} : H^s(\mathbb{R}) \longrightarrow C([0, T]; H^s(\mathbb{R}))$ is C^2 at the origin. When $\phi \in H^s(\mathbb{R})$, we will denote $u_\phi(t) = S^{j,k}(t)\phi$ the solution of the Cauchy problem (1) with initial data ϕ. This means that u_ϕ is a solution of the integral equation

$$u(t) := U_j(t)u_0 + \int_0^t U_j(t-t')\partial_x^k(u^2)dt'. \tag{117}$$

When ϕ and ψ are in $H^s(\mathbb{R})$, we use the fact that the nonlinearity $\partial_x^k(uv)$ is a bilinear symmetric application to compute the Fréchet derivative of $S^{j,k}(t)$ at ψ in the direction ϕ

$$d_\psi S^{j,k}(t)\phi = U_j(t)\phi + 2\int_0^t U_j(t-t')\partial_x^k(u_\psi(t')d_\psi S^{j,k}(t')\phi)dt'. \tag{118}$$

Since the Cauchy problem (1) is supposed to be well-posed, we know using the uniqueness that $S^{j,k}(t)0 = u_0(t) = 0$ and then we deduce from (118) that

$$d_0 S^{j,k}(t)\phi = U_j(t)\phi. \tag{119}$$

Using (118), we compute the second Fréchet derivative at the origin in the direction (ϕ, ψ)

$$d_0^2 S^{j,k}(t)(\phi, \psi) = d_0(d\, S^{j,k}(t)\phi)\psi = \frac{\partial}{\partial_\beta}(\beta \mapsto d_{\beta\psi} S^{j,k}(t)\phi)_{|\beta=0}$$

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$$= 2 \int_0^t U_j(t - t')\partial_x^k (d_{\beta\psi} S^{j,k}(t')\psi d_{\beta\psi} S^{j,k}(t')\phi)dt'_{|\beta=0}$$

$$+ 2 \int_0^t U_j(t - t')\partial_x^k (u_{\beta\psi}(t')d_{\beta\psi}^2 S^{j,k}(t')(\phi, \psi))dt'_{|\beta=0}.$$

Thus we deduce using (119) that

$$d_0^2 S^{j,k}(t)(\phi, \psi) = 2 \int_0^t U_j(t - t')\partial_x^k [(U_j(t')\psi)(U_j(t')\phi)]dt'. \qquad (120)$$

The assumption of C^2 regularity of $S^{j,k}(t)$ at the origin would imply that $d_0^2 S^{j,k}(t) \in \mathcal{B}(H^s(\mathbb{R}) \times H^s(\mathbb{R}), H^s(\mathbb{R}))$, which would lead to the following inequality

$$\|d_0^2 S^{j,k}(t)(\phi, \psi)\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R})}\|\psi\|_{H^s(\mathbb{R})}, \quad \forall \, \phi, \, \psi \in H^s(\mathbb{R}). \qquad (121)$$

But (121) is equivalent to (106) which has been shown to fail in the proof of Theorem 3. $\qquad \square$

The case of the higher-order Benjamin-Ono and intermediate long wave equations. In order to study the Cauchy problems (3) (respectively (4)), we define V_1 (respectively $V_2(t)$) the unitary group in $H^s(\mathbb{R})$ associated to the linear part of the equations, *i.e.*

$$V_k(t)\phi = \left(e^{ip_k(\xi)t}\widehat{\phi} \right)^{\vee}, \quad k = 1, 2, \quad \forall \, t \in \mathbb{R}, \quad \forall \phi \in H^s(\mathbb{R}), \qquad (122)$$

where

$$p_1(\xi) = b|\xi|\xi + a\epsilon\xi^3,$$

and

$$p_2(\xi) = b \coth(h\xi)\xi^2 + (a_1 \coth^2(h\xi) + a_2)\epsilon\xi^3.$$

We denote by f_1 (respectively f_2) the nonlinearity of the equations (3) (respectively (4)), *i.e.*

$$f_1(u) = cu\partial_x u - d\epsilon\partial_x(uH\partial_x u + H(u\partial_x u)),$$

and

$$f_2(u) = cu\partial_x u - d\epsilon\partial_x(u\mathcal{F}_h\partial_x u + \mathcal{F}_h(u\partial_x u)).$$

Then, we have the analogous of Theorem 3 for the equations (3) and (4).

Theorem 6 *Let $s \in \mathbb{R}$, $T > 0$ and $k \in \{1, 2\}$. Then, there does not exist any space X_T such that X_T is continuously embedded in $C([-T, T]; H^s(\mathbb{R}))$, i.e.,*

$$\|u\|_{C([-T,T];H^s)} \lesssim \|u\|_{X_T}, \quad \forall \, u \in X_T, \tag{123}$$

and such that

$$\|V_k(t)\phi\|_{X_T} \lesssim \|\phi\|_{H^s}, \quad \forall \, \phi \in H^s(\mathbb{R}), \tag{124}$$

and

$$\| \int_0^t V_k(t - t') f_k(u)(t') dt' \|_{X_T} \lesssim \|u\|_{X_T}^2, \quad \forall \, u \in X_T. \tag{125}$$

Theorem 5 is a consequence of Theorem 6 (see the proof of Theorem 4).

Proof of Theorem 6. Let $s \in \mathbb{R}$, $T > 0$ and $k \in \{1, 2\}$. Suppose that there exists a space X_T such as in Theorem 6. Take $\phi \in H^s(\mathbb{R})$, and define $u(t) = V_k(t)\phi$. Then, we use (123), (124) and (125) to see that

$$\| \int_0^t V_k(t - t') f_k(V_k(t'))\phi) dt' \|_{H^s} \lesssim \|\phi\|_{H^s}^2. \tag{126}$$

We will show that (126) fails for an appropriate choice of ϕ, which would lead to a contradiction.

Define ϕ by [1]

$$\phi = \left(\alpha^{-1/2} \chi_{I_1} + \alpha^{-1/2} N^{-s} \chi_{I_2} \right)^{\vee} \tag{127}$$

where

$$N \gg 1, \quad 0 < \alpha \ll 1, \quad I_1 = [\alpha/2, \alpha] \quad \text{and} \quad I_2 = [N, N + \alpha] \tag{128}$$

Note first that

$$\|\phi\|_{H^s} \sim 1. \tag{129}$$

Then, the same computation as for (128) leads to

$$\left(\int_0^t V_k(t - t') f_k((V_k(t')\phi) dt' \right)^{\wedge} (\xi) \sim g_1(\xi, t) + g_2(\xi, t) + g_3(\xi, t), \tag{130}$$

where,

$$g_1(\xi, t) = \frac{e^{itp(\xi)}}{\alpha} \int_{\substack{\xi_1 \in I_1 \\ \xi - \xi_1 \in I_1}} \tilde{f}_k(\xi, \xi_1) \frac{e^{it(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} - 1}{i(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} d\xi_1,$$

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[1] We can also take $\operatorname{Re}\phi$ instead of ϕ (see the remark after the proof of Theorem 3).

$$g_2(\xi, t) = \frac{e^{itp(\xi)}}{\alpha N^{2s}} \int_{\substack{\xi_1 \in I_2 \\ \xi - \xi_1 \in I_2}} \tilde{f}_k(\xi, \xi_1) \frac{e^{it(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} - 1}{i(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} d\xi_1,$$

$$g_3(\xi, t) = \frac{e^{itp(\xi)}}{\alpha N^s} \left(\int_{\substack{\xi_1 \in I_1 \\ \xi - \xi_1 \in I_2}} \tilde{f}_k(\xi, \xi_1) \frac{e^{it(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} - 1}{i(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} d\xi_1 \right.$$

$$\left. + \int_{\substack{\xi_1 \in I_2 \\ \xi - \xi_1 \in I_1}} \tilde{f}_k(\xi, \xi_1) \frac{e^{it(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} - 1}{i(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} d\xi_1 \right),$$

and
$$\tilde{f}_1(\xi, \xi_1) = c\xi_1 - d\epsilon(\xi |\xi_1| + |\xi|\xi_1),$$

or
$$\tilde{f}_2(\xi, \xi_1) = c\xi_1 - d\epsilon(\xi \coth(\xi_1)\xi_1 + \coth(\xi)\xi\xi_1).$$

Since the supports of $g_1(\cdot, t)$, $g_2(\cdot, t)$ and $g_3(\cdot, t)$ are disjoint, we use (130) to bound by below the left-hand side of (126)

$$\| \int_0^t V_k(t - t') f_k((V_k(t'))\phi) dt' \|_{H^s} \geq \|(g_3)^{\vee}(\xi, t)\|_{H^s}. \tag{131}$$

We notice that the function p_k is smooth and that

$$|p_k'(\xi)| \lesssim 1 + |\xi|^2. \tag{132}$$

Thus, when $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_2$ or $\xi - \xi_1 \in I_1$ and $\xi_1 \in I_2$, we have that $|\xi| \sim N$, and we use (132) and the mean value theorem to get the estimate

$$|p(\xi_1) + p(\xi - \xi_1) - p(\xi)| \lesssim \alpha N^2. \tag{133}$$

Hence we choose $\alpha = N^{-2-\epsilon}$, with $0 < \epsilon < 1$, to get

$$\left| \frac{e^{it(p(\xi_1) + p(\xi - \xi_1) - p(\xi))} - 1}{p(\xi_1) + p(\xi - \xi_1) - p(\xi)} \right| = |t| + o(N^{-\epsilon}). \tag{134}$$

We are now able to give a lower bound for $\|(g_3)^{\vee}(\xi, t)\|_{H^s}$

$$\|(g_3)^{\vee}(\xi, t)\|_{H^s} \gtrsim \frac{N^s}{N^s \alpha} \left(N^2 \alpha^{1/2} \alpha - N \alpha \alpha^{1/2} \alpha \right) \gtrsim N^2 \alpha^{1/2}. \tag{135}$$

Thus, we conclude from (126), (129), (131) and (135) that

$$N^2 \alpha^{1/2} = N^{1-\epsilon/2} \lesssim 1, \quad \forall N \gg 1, \tag{136}$$

which is a contradiction. $\qquad \square$

References

[1] C. R. Argento, *O problema de Cauchy para a equação de Kuramoto-Velarde generalizada com dispersão*, Tese de doutorado, IMPA, (1997).

[2] J. Bourgain, *Periodic Korteweg de Vries equation with measures as initial data,* Sel. Math. New. Ser. **3** (1997), 115-159.

[3] W. Craig, P. Guyenne, H. Kalisch, *Hamiltonian long wave expansions for free surfaces and interfaces,* Comm. Pure Appl. Math. **58** (2005), 1587-1641.

[4] M. Christ, A. Kiselev, *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001) 406-425.

[5] C. E. Kenig, G. Ponce, L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991) 33-69.

[6] C. E. Kenig, G. Ponce, L. Vega, *Small solutions to nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré Anal. Non Linaire **10** (1993) 255-288.

[7] C. E. Kenig, G. Ponce, L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993) 527-620.

[8] C. E. Kenig, G. Ponce, L. Vega, *On the hierarchy of the generalized KdV equations*, Proc. Lyon Workshop on singular limits of dispersive waves **320** (1994), 347-356.

[9] C. E. Kenig, G. Ponce, L. Vega, *Higher-order nonlinear dispersive equations*, Proc. Amer. Math. Soc. **122** (1994) 157-166.

[10] C. E. Kenig, A. Ruiz, *A strong type (2,2) estimate for a maximal operator associated to the Schrödinger equation*, Trans. Amer. Math. Soc. **280** (1983) 239-246.

[11] L. Molinet, J.C. Saut, N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, SIAM J. Math. Anal. Vol.33, No. 4 (2001), 982-988.

[12] L. Molinet, J.C. Saut, N. Tzvetkov *Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation*, Duke Math J. **115** (2002), 353-384.

[13] L. Molinet, F. Ribaud, *On the Cauchy problem for the generalized Korteweg-de Vries equation*, Comm. Part. Diff. Equa. **28** (2003), 2065-2091.

[14] L. Molinet, F. Ribaud, *Well-posedness results for the generalized Benjamin-Ono equation with small initial data*, J. Math. Pures Appl. **83** (2004), 277-311.

[15] D. Pilod, *The Cauchy problem for the dispersive Kuramoto-Velarde equation*, Tese de doutorado, IMPA, (2006).

[16] F. Planchon, *Dispersive estimates and the 2D cubic Schrödinger equation*, J. Math. Anal. Math. **79** (2000), 809-820.

[17] G. Ponce, *Regularity of solutions to nonlinear dispersive equations*, J. Diff. Equations **78** (1989), 122-135.

[18] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J. 1970.

[19] H. Triebel, *Theory of function spaces,* Monographs in Mathematics, vol. 17, Birkhauser, 1983.

[20] N. Tzvetkov, *Remark on the local ill-posedness for the KdV equation*, C.R. Acad. Sci. Paris Ser.I Math., **329** (1999), 1043-1047.

UFRJ, Institute of Mathematics,
P.O. Box 68530 - Cidade Universitária.
Ilha do Fundão. CEP 21945-970
Rio de Janeiro, RJ, Brazil.
E-mail: pilod@impa.br, didier@im.ufrj.br